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BY D. L. WEBSTER.

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In the science of mechanics of ordinary matter we are accustomed to regard velocity as essentially relative but acceleration as absolute; and to say that, if a body is not acted upon in any way by other bodies, its acceleration is zero, but that, if it is acted upon by any other body, the accelerations of the two are opposite, and inversely proportional to their masses. But how can we test this law? and how can we measure the acceleration? If we measure the velocity relative to the earth, or to the sun, or to any star, at any two times separated by a very short interval, how can we be sure that the system of reference has not been accelerated during the time that has elapsed? And if it has, on what system is its acceleration measured?

This difficulty is made still more puzzling if we consider two mechanical systems, such as the solar system, exactly similar in every way, but one of which is removed to a practically infinite distance from all other matter while the other is subject to the attraction of a tremendous mass, so large and far removed that its gravitational field is practically uniform, and at absolute zero temperature so that no radiation would be received from it. These systems would be accelerated relatively to each other, but which of them would be accelerated? No observer on either of them could tell by any mechanical means.

An answer to these questions appears to be given by the electromagnetic equations, which assume the especially simple form with which we are familiar when expressed in terms of the length, mass, and time units of any one of a certain set of systems, any one of which appears to be moving relative to any other with a constant velocity, less than the velocity of light. These systems may all be assumed to be unaccelerated, and assuming the impossibility of any system's moving relative to one of these with a velocity greater than that of light, we say that all other systems are accelerated.

But acceleration means rate of change of velocity, and therefore absolute acceleration means rate of change of absolute velocity, and, if there is no such thing as absolute velocity, how can there be such a thing as rate of change of it? We must, therefore, redefine the absolute acceleration of any system to mean the acceleration relative to another system moving in such a way that the simple electromagnetic equations hold on it, and on which the velocity of the first system is zero at the instant in question.

Even now, however, there is difficulty, because in place of our set of systems with the constant relative velocities, for which the equations hold, we might equally well imagine any other exactly similar set of systems each of which has a certain given acceleration relative to the corresponding one of the first set. And, disregarding the rather arbitrary definition of absolute acceleration given above, it is evident that, if space had no properties other than those of geometry and time, any difference between the laws of nature as observed from two of these relatively accelerated systems would be impossible. But since the observed laws are simpler in one of the first set than in one accelerated relative to it, the space must have other properties than the above mentioned ones; and, because of these properties, it appears highly probable that there must be some sort of a substance, or medium, filling all space, having no acceleration relative to any of the systems for which the simple electromagnetic equations hold, not directly affecting our senses, but having properties which account for all the laws of the phenomena that are directly observable, including the exact mathematical similarity of the expressions for these laws in terms of quantities measured on any system moving with uniform velocity, less than that of light, through it. This is the medium to which we give the name of "ether."

The Ether.—To obtain any knowledge of the properties of this medium, that enable it to show the phenomena of electricity, magnetism, and gravitation, and to account for the laws of motion of matter, the principal of relativity, and the permanent existence of positive and negative electrons in spite of the possibility of collisions between them, it will be necessary to obtain the simplest possible form of the set of laws which govern these phenomena.

Since many of the quantities that we deal with are vectors, we shall find it convenient to use some simple vector analysis, with the following notation, that of Gibbs, in which all vectors will be printed in Clarendon type while scalars are in italic type. The scalar product,

$$(\mathbf{a}_x\mathbf{b}_x + \mathbf{a}_y\mathbf{b}_y + \mathbf{a}_z\mathbf{b}_z),$$

of any two vectors, $\mathbf{a}$ and $\mathbf{b}$, will be denoted by $\mathbf{a}\cdot\mathbf{b}$; while the vector product,

$$\mathbf{i}\,(a_y b_z - a_z b_y) + \mathbf{j}\,(a_z b_x - a_x b_z) + \mathbf{k}\,(a_x b_y - a_y b_x),$$

will be denoted by $\mathbf{a}\times\mathbf{b}$, where $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the unit vectors in the directions of x, y, and z respectively.

The symbol ∇ will be used for the operator,

$$\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right),$$

so that ∇a = the gradient of scalar a, a vector; $\nabla\cdot\mathbf{a}$ = the divergence of vector $\mathbf{a}$, a scalar; and $\nabla\times\mathbf{a}$ = the curl of vector $\mathbf{a}$, another vector.

The symbol "Pot" will be used for the operation of taking the Newtonian potential of any function, so that

$$\text{Pot}\, a = \int\!\!\int\!\!\int_{\infty} \frac{a}{r}\, d\tau,$$

where r is the distance from $d\tau$ to the point at which we wish to find Pot a. We may apply the operator Pot to a vector as well as a scalar, and, in either case, Poisson's equation tells us that

$$(\nabla^2\,\text{Pot}) = -4\pi,$$

or the application of the operator ∇^2 to the Pot of any function gives -4π times the original function.

It will be found convenient to indicate differentiation with respect to ct, where c is the velocity of light, by a dot over the letter that stands for the function. Thus

$$\dot{\mathbf{a}} = \frac{\partial \mathbf{a}}{\partial (ct)}.$$

For brevity let us assume also, unless otherwise stated, that the functions used in the following work all vanish at infinity and are finite and continuous throughout space.

Laws of the Ether.—To write out the laws of the ether in the form that accounts for all the above mentioned phenomena, we must distinguish between the effects due to positive and negative charges, and, therefore, it will be convenient to call the density of positive electricity at any point $\overset{+}{\rho}$, (a quantity which is always positive), and that of

negative electricity $\bar{\rho}$, (always negative). The electric forces due to all the positive and all the negative electricity we may call $\overset{+}{\mathbf{E}}$ and $\bar{\mathbf{E}}$, and the magnetic forces $\overset{+}{\mathbf{H}}$ and $\bar{\mathbf{H}}$, while the velocities of the charges may be represented in terms of $\overset{+}{\boldsymbol{\beta}}$ and $\bar{\boldsymbol{\beta}}$, their ratios to the velocity of light.

These quantities may be supposed to satisfy the following set of equations:

$$(1)\quad \nabla\cdot\overset{+}{\mathbf{E}} = \overset{+}{\rho}, \qquad (2)\quad \nabla\cdot\bar{\mathbf{E}} = \bar{\rho},$$

$$(3)\quad \nabla\times\overset{+}{\mathbf{H}} = \dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}, \qquad (4)\quad \nabla\times\bar{\mathbf{H}} = \dot{\bar{\mathbf{E}}} + \bar{\rho}\bar{\boldsymbol{\beta}},$$

$$(5)\quad \nabla\cdot\overset{+}{\mathbf{H}} = 0, \qquad (6)\quad \nabla\cdot\bar{\mathbf{H}} = 0,$$

$$(7)\quad \nabla\times\overset{+}{\mathbf{E}} = -\dot{\overset{+}{\mathbf{H}}}, \qquad (8)\quad \nabla\times\bar{\mathbf{E}} = -\dot{\bar{\mathbf{H}}},$$

which determine the positive forces in terms of the positions and velocities of the positive charges, and the negative forces in terms of those of the negative charges. But in addition to these equations we have the following pair,

$$(9)\quad \mathbf{E} + \overset{+}{\boldsymbol{\beta}}\times\mathbf{H} + \overset{+}{\mathbf{K}} - G(\overset{+}{\mathbf{E}} + \overset{+}{\boldsymbol{\beta}}\times\overset{+}{\mathbf{H}}) = 0,$$

$$(10)\quad \mathbf{E} + \bar{\boldsymbol{\beta}}\times\mathbf{H} + \bar{\mathbf{K}} = 0,$$

which must hold at every point of every electron, positive or negative, and in which $\overset{+}{\mathbf{K}}$ and $\bar{\mathbf{K}}$ are forces per unit charge due to the internal stresses of the electron, while G is a very small number whose presence in equation (9) accounts for the phenomena of gravitation.[1]

The laws governing the vectors $\overset{+}{\mathbf{K}}$ and $\bar{\mathbf{K}}$ may be deduced from the fact that the deformation of the electron when its velocity is very great is the same as that of a perfectly flexible and compressible, charged, conducting shell, with no internal stresses, subject to a constant external hydrostatic pressure or internal hydrostatic tension.[2] Therefore, we may assume that $\overset{+}{\mathbf{K}}$ and $\bar{\mathbf{K}}$ are forces such as would result from such a tension, and that they are transmitted by

1 For further details on this point see D. L. Webster, "On an Electromagnetic Theory of Gravitation," These Proceedings, **47**, 14 (1912). The reasoning and conclusions are changed but little if we introduce a similar term in equation (10), and thereby gain in symmetry in our theory.

2 Poincaré, Comptes Rendus, **140**, 1504–8.

the material of the electron. This tension is, of course, constant throughout its volume only if all the charge of the electron is on its surface, otherwise, it increases as we go nearer the centre of the electron.

Abraham has raised the objection to this theory, that it involves an instability of the shape of the electron,[3] that would soon destroy all such bodies. But this objection is based on the interpretation of the above vectors as mechanical forces per unit charge, tending to accelerate the parts of the electrons involved, and on the idea that a part of the charge may in some way be displaced from the position in which all these forces exactly balance. Such displacements would result in a rapid disruption of the electron, a process in which equations (9) or (10) could not hold indefinitely. But if we take them as expressions of a fundamental law, which would be violated by such a process, we have a reason why this process cannot occur, nor even start to occur, and the problem of stability of shape of the election is solved.

To determine the motion of a whole electron from these equations (9) and (10) we are aided by the fact that the resultant of the internal forces is zero, but we have to remember that the vectors $\overset{+}{\mathbf{E}}$, $\overset{-}{\mathbf{E}}$, $\overset{+}{\mathbf{H}}$, and $\overset{-}{\mathbf{H}}$, that occur in these equations, include not only the contributions from external sources, but internal as well. Therefore, the equations demand motion with constant velocity when the external forces are zero, and the resultant of the actions of different parts of the electron on each other must be zero also. But if the external forces are not zero, each part of the electron must be accelerated in such a manner that the resultant of all the forces, radiated or otherwise, from all other parts, will just balance the resultant of the external forces. Thus we have a reason for the apparent inertia of every electron, and of bodies composed of electrons, so that the laws of motion of matter may be proved to be consequences of the laws of electromagnetic forces.

Simplification of the Laws. Hamilton's Principle.

We may, however, simplify these laws still further, by remembering the fact that there is one dynamical principle that applies to all motions of matter and also to all the phenomena of slow changes of positions of electric charges and of the positions and magnitudes

[3] See Lorentz, Theory of Electrons; Chap. V, 1905.

of currents, and expresses the laws of the phenomena perfectly with no other assumptions than equations (1)–(4). Therefore, it seems reasonable to suppose that this same one principle may replace equations (5)–(10), and reduce the number of necessary laws from 10 to 5.

This fundamental principle is Hamilton's Principle, which says that for any dynamical system whose kinetic and potential energies are T and W respectively,

$$\delta \int_{t_1}^{t_2} (T - W)\, dt = 0,$$

where t_1 and t_2 are any two times, and where the variation from the actual motion is any variation, consistent with the constraints of the system, that makes the configurations of the system at the times t_1 and t_2 the same as it is in the actual motion. In the case of the ether, writing $\mathbf{E}$ for $\overset{+}{\mathbf{E}} + \overset{-}{\mathbf{E}}$ and $\mathbf{H}$ for $\overset{+}{\mathbf{H}} + \overset{-}{\mathbf{H}}$, this principle takes the form,

$$(11) \quad \delta \int_{t_1}^{t_2} \int_{\infty} \{(\mathbf{H}^2 - G\overset{+}{\mathbf{H}}^2) - (\mathbf{E}^2 - G\overset{+}{\mathbf{E}}^2 + 2U)\}\, d\tau dt = 0,$$

where U is the sum of the hydrostatic tensions in the positive and negative electrons, if any, in which the element $d\tau$ lies, and which produce the forces $\overset{+}{\mathbf{K}}$ and $\overset{-}{\mathbf{K}}$, and where two configurations are to be considered the same if, and only if, the vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$ are the same in one as in the other.[4]

To prove that equations (5)–(6) result from equations (1)–(4) and equation (11) we may write (11) in the form,

$$(12) \quad \int_{t_1}^{t_2} \int_{\infty} \{(\mathbf{H}\cdot\delta\mathbf{H} - G\overset{+}{\mathbf{H}}\cdot\delta\overset{+}{\mathbf{H}}) - (\mathbf{E}\cdot\delta\mathbf{E} - G\overset{+}{\mathbf{E}}\cdot\delta\overset{+}{\mathbf{E}} + \delta U)\}\, d\tau dt = 0,$$

and then suppose that $\delta\mathbf{H}$, $\delta\mathbf{E}$, $\delta\overset{+}{\mathbf{E}}$, and δU are all zero throughout the interval. Now whatever vector $\overset{+}{\mathbf{H}}$ may be we may split it into a sum of two parts, $\overset{+}{\mathbf{H}}_S$ and $\overset{+}{\mathbf{H}}_L$, such that

$$\nabla\cdot\overset{+}{\mathbf{H}}_S = 0 = \nabla\times\overset{+}{\mathbf{H}}_L,$$

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4 For another form of Hamilton's Principle, involving different assumptions see Larmor, "Aether and Matter," Chapter I.

and then write

$$\int\int\int \overset{+}{\mathbf{H}}^2 d\tau$$

as

$$\int\int\int (\overset{+}{\mathbf{H}}_S{}^2 + 2\,\overset{+}{\mathbf{H}}_S \cdot \overset{+}{\mathbf{H}}_L + \overset{+}{\mathbf{H}}_L{}^2)\, d\tau.$$

But by Green's Theorem, whatever these parts are, if both vanish at infinity,

$$\int\int\int \overset{+}{\mathbf{H}}_S \cdot \overset{+}{\mathbf{H}}_L d\tau = 0. \tag{13}$$

In this case $\overset{+}{\mathbf{H}}_S$ is completely determined by equation (3), so that, if $\delta\mathbf{E}$, $\delta\overset{+}{\mathbf{E}}$, and δU are zero, $\delta\overset{+}{\mathbf{H}}_S$ is zero, therefore

$$\begin{aligned}\int\int\int \delta\overset{+}{\mathbf{H}}^2 d\tau &= 2\int\int\int (\overset{+}{\mathbf{H}}_S \cdot \delta\overset{+}{\mathbf{H}}_L + \overset{+}{\mathbf{H}}_L \cdot \delta\overset{+}{\mathbf{H}}_L)\, d\tau \\ &= 2\int\int\int \overset{+}{\mathbf{H}}_L \cdot \delta\overset{+}{\mathbf{H}}_L d\tau.\end{aligned} \tag{14}$$

But this must be zero whatever $\delta\overset{+}{\mathbf{H}}_L$ is, therefore $\overset{+}{\mathbf{H}}_L$ is zero, as is $\overset{-}{\mathbf{H}}_L$ also. Therefore

$$\nabla \cdot \overset{+}{\mathbf{H}} = 0, \tag{5}$$

$$\nabla \cdot \overset{-}{\mathbf{H}} = 0. \tag{6}$$

To derive equations (7) and (8) we may introduce vectors $\overset{+}{\mathbf{I}}$, $\overset{-}{\mathbf{I}}$, $\overset{+}{\mathbf{P}}$, and $\overset{-}{\mathbf{P}}$, defined by the equations,

$$\overset{+}{\mathbf{I}} = \int_0^{ct} \overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}\, d(ct), \qquad \overset{-}{\mathbf{I}} = \int_0^{ct} \overset{-}{\rho}\overset{-}{\boldsymbol{\beta}}\, d(ct), \tag{15}$$

$$\overset{+}{\mathbf{P}} = \frac{1}{4\pi}\,\mathrm{Pot}\,(\overset{+}{\mathbf{E}} + \overset{+}{\mathbf{I}}), \qquad \overset{-}{\mathbf{P}} = \frac{1}{4\pi}\,\mathrm{Pot}\,(\overset{-}{\mathbf{E}} + \overset{-}{\mathbf{I}}). \tag{16}$$

From these equations we may infer

$$\nabla^2\overset{+}{\mathbf{P}} = -(\overset{+}{\mathbf{E}} + \overset{+}{\mathbf{I}}), \tag{17}$$

$$\nabla^2\overset{\cdot}{\overset{+}{\mathbf{P}}} = -(\overset{\cdot}{\overset{+}{\mathbf{E}}} + \overset{\cdot}{\overset{+}{\mathbf{I}}}) = -(\overset{\cdot}{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}).$$

From equation (3) we know that

$$\nabla \cdot (\overset{\cdot}{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}) = 0, \tag{18}$$

so that $$\frac{1}{4\pi}\nabla\cdot\mathrm{Pot}\,(\overset{+}{\mathbf{E}}+\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}})=\nabla\cdot\overset{+}{\mathbf{P}}=0;$$

and since, whatever $\overset{+}{\mathbf{P}}$ is,

$$\nabla\times\nabla\times\overset{+}{\mathbf{P}}=-\nabla^2\overset{+}{\mathbf{P}}+\nabla(\nabla\cdot\overset{+}{\mathbf{P}}),$$

$$\nabla\times\nabla\times\overset{+}{\mathbf{P}}=(\overset{+}{\mathbf{E}}+\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}})=\nabla\times\overset{+}{\mathbf{H}}, \tag{19}$$

which, combined with (5), gives

$$\nabla\times\overset{+}{\mathbf{P}}=\overset{+}{\mathbf{H}}. \tag{20}$$

Similarly $$\nabla\times\overset{-}{\mathbf{P}}=\overset{-}{\mathbf{H}}. \tag{21}$$

Equation (12) now takes the form

$$\int_{t_1}^{t_2}\int_{\infty}\{(\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\dot{\mathbf{P}}-G\nabla\times\overset{+}{\dot{\mathbf{P}}}\cdot\nabla\times\delta\overset{+}{\dot{\mathbf{P}}})-\tfrac{1}{2}\delta(\mathbf{E}^2-G\overset{+}{\mathbf{E}}^2+2U)\}d\tau dt=0. \tag{22}$$

But

$$\int_{t_1}^{t_2}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\dot{\mathbf{P}}dt=\frac{1}{c}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\nabla\times\ddot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}dt, \tag{23}$$

$$=\frac{1}{c}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}[\nabla\cdot\{\ddot{\mathbf{P}}\times(\nabla\times\delta\mathbf{P})\}+\ddot{\mathbf{P}}\cdot\nabla\times\nabla\times\delta\mathbf{P}]dt.$$

$\therefore$

$$\int_{t_1}^{t_2}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\dot{\mathbf{P}}dt=\frac{1}{c}\Big[\int_{\infty}\nabla\times\dot{\mathbf{P}}\cdot\nabla\times\delta\mathbf{P}d\tau\Big]_{t_1}^{t_2}-\int_{t_1}^{t_2}\int_{\infty}\ddot{\mathbf{P}}\cdot\nabla\times\nabla\times\delta\mathbf{P}d\tau dt-\int_{t_1}^{t_2}\lim_{S=\infty}\int_{s}\ddot{\mathbf{P}}\times(\nabla\times\delta\mathbf{P})\cdot d\mathbf{S}dt, \tag{24}$$

where S is any closed surface that may recede indefinitely in all directions from any interior point, and of which $d\mathbf{S}$ is an element considered as a vector in the direction of the exterior normal. If we now let $\delta\mathbf{I}.=0$, then $\delta\mathbf{P}=0$ at t_1 and t_2, and the first term on the right side of equation (24) drops out, and so does the surface integral when there is only a finite amount of charge in the universe.

Treating $\overset{+}{\mathbf{P}}$ in the same way, we obtain, if

$$\delta\overset{+}{\mathbf{I}}=\delta\overset{-}{\mathbf{I}}=0,$$

(25)

$$\int_{t_1}^{t_2}\{(\ddot{\mathbf{P}}\cdot\nabla\times\nabla\times\delta\mathbf{P}-G\ddot{\dot{\mathbf{P}}}\cdot\nabla\times\nabla\times\delta\mathbf{P})+(\mathbf{E}\cdot\delta\mathbf{E}-G\dot{\mathbf{E}}\cdot\delta\dot{\mathbf{E}}+\delta U)\}d\tau dt=0,$$

or, since $\delta U = 0$ and

(26) $$\nabla\times\nabla\times\delta\mathbf{P} = -\nabla^2\delta\mathbf{P} = +\delta\mathbf{E},$$

when no motions of charges have been varied,

(27) $$\int_{t_1}^{t_2}\int_{\infty}\{(\ddot{\mathbf{P}}+\mathbf{E})\cdot\delta\mathbf{E}-G(\ddot{\dot{\mathbf{P}}}+\dot{\mathbf{E}})\cdot\delta\dot{\mathbf{E}}\}\,d\tau dt=0.$$

Splitting $\mathbf{E}$ into the parts $\mathbf{E}_S$ and $\mathbf{E}_L$, treating $\dot{\mathbf{E}}$ likewise, and applying Green's Theorem as in (14),

(28) $$\int_{t_1}^{t_2}\int_{\infty}\{(\ddot{\mathbf{P}}+\mathbf{E}_S)\cdot\delta\mathbf{E}_S-G(\ddot{\dot{\mathbf{P}}}-\dot{\mathbf{E}}_S)\cdot\delta\dot{\mathbf{E}}_S\}\,d\tau kt=0,$$

because $\delta\mathbf{E}_L$ and $\delta\dot{\mathbf{E}}_L$ are zero when no charge motions are varied.

(29) Therefore $$\ddot{\dot{\mathbf{P}}}+\dot{\mathbf{E}}_S=0,$$

(30) and $$\ddot{\dot{\mathbf{P}}}+\dot{\bar{\mathbf{E}}}_S=0,$$

(31) or $$\dot{\mathbf{E}}_S=-\frac{1}{4\pi}\,\mathrm{Pot}(\ddot{\dot{\mathbf{E}}}+\dot{\dot{\rho}}\,\dot{\boldsymbol{\beta}}).$$

(32) and $$\dot{\bar{\mathbf{E}}}_S=-\frac{1}{4\pi}\,\mathrm{Pot}(\ddot{\dot{\mathbf{E}}}+\dot{\bar{\rho}}\,\dot{\boldsymbol{\beta}}).$$

Applying $\nabla\times$ to (29) and (30) we have

(7) $$\nabla\times\dot{\mathbf{E}} = -\dot{\mathbf{H}},$$ (8) $$\nabla\times\dot{\bar{\mathbf{E}}} = -\dot{\bar{\mathbf{H}}}.$$

To derive equations (9) and (10) from equation (11) or (12) let us suppose that, for a short time during the interval t_1 and t_2, an *infinitesimal* positive charge de, occupying a small tube of length $d\mathbf{r}'$ and cross section $d\sigma$ [5] and lying in the direction of $\dot{\boldsymbol{\beta}}$ is displaced in some other direction through a distance $\delta\mathbf{r}$. To satisfy equation (1) with this variation we may superpose on the actual value of $\dot{\mathbf{E}}$ a straight

[5] Any element of surface may be considered as a vector along its normal, and when its direction is chosen, the positive direction around its boundary is that of a right-handed screw rotation.

tube of the vector $\delta\overset{+}{\mathbf{E}}$ connecting the new position of de to the old, the flux of $\delta\overset{+}{\mathbf{E}}$ being against the direction of $\delta\mathbf{r}$ and of magnitude de. And to satisfy equations (3) and (5) we must also assume a certain variation $\delta\overset{+}{\mathbf{H}}$ which is uniquely defined by these equations and the variations assumed above. We may now assume no variations of the negative forces, and for the positive forces only the necessary variation of $\overset{+}{U}$ and the variations specified above.

With these assumptions (12) becomes

$$(33)\quad \int_{t_1}^{t_2}\int\int_{\infty}\int\{(\mathbf{H}-G\overset{+}{\mathbf{H}})\cdot\delta\overset{+}{\mathbf{H}}-(\mathbf{E}-G\overset{+}{\mathbf{E}})\cdot\delta\overset{+}{\mathbf{E}}-\delta\overset{+}{U}\}d\tau dt=0.$$

(34) To calculate

$$-\int\int_{\infty}\int(\mathbf{E}-G\overset{+}{\mathbf{E}})\cdot\delta\overset{+}{\mathbf{E}}d\tau,$$

we need to integrate only over the tube of $\delta\overset{+}{\mathbf{E}}$ defined above, so that, if de is of infinitesimal size, we may take for the result,

$$(35)\qquad (\mathbf{E}-G\overset{+}{\mathbf{E}})\cdot\delta\mathbf{r}de.$$

To calculate

$$(36)\qquad \int\int_{\infty}\int(\mathbf{H}-G\overset{+}{\mathbf{H}})\cdot\delta\overset{+}{\mathbf{H}}d\tau,$$

we may consider $\delta\overset{+}{\mathbf{H}}$ as the $\overset{+}{\mathbf{H}}$ produced by a current of strength $\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}\cdot d\sigma$ flowing around the edge of the parallelogram one side of which contains the old position of de and the other side the new position, while the remaining sides are the tubes of the vector $\delta\overset{+}{\mathbf{E}}$ made necessary by the motion of the parallelogram. We may now evaluate the integral, splitting the space up into elements of each of which two sides, $d\mathbf{S}$, are surfaces whose normals are in the direction of $\delta\overset{+}{\mathbf{H}}$ while the remaining dimension, $d\mathbf{s}$, is in the same direction. The integral may now be written

$$(37)\qquad \int\int_{\infty}\int(\mathbf{H}-G\overset{+}{\mathbf{H}})\cdot\delta\overset{+}{\mathbf{H}}d\mathbf{S}\cdot d\mathbf{s},$$

(38) or $$\int\int_{\infty}\int (\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot d\mathbf{S}\delta\overset{+}{\mathbf{H}}\cdot d\mathbf{s}.$$

But since $$\nabla\cdot\mathbf{H} = \nabla\cdot\overset{+}{\mathbf{H}} = 0,$$

the surface integral

(39) $$\int\int (\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot d\mathbf{S}$$

is the same over any cap of the parallelogram circuit as over any other; and since

(40) $$\nabla\times\delta\overset{+}{\mathbf{H}} = \delta\dot{\overset{+}{\mathbf{E}}} + \delta(\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}),$$

the line integral

(41) $$\int \delta\overset{+}{\mathbf{H}}\cdot d\mathbf{s}$$

is the same around any line of the vector $\delta\overset{+}{\mathbf{H}}$ as on any other. Therefore the integral (38) is

(42) $$\Big[\int\int_{\text{Any cap}} (\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot d\mathbf{S}\Big]\Big[\int_{\text{Any line}} \delta\overset{+}{\mathbf{H}}\cdot d\mathbf{s}\Big].$$

But by Stokes' Theorem, the line integral is

(43) $$\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}\cdot d\sigma,$$

while the surface integral over the plane cap is,

(44) $$d\mathbf{r}'\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r},$$

so that (36) becomes

(45) $$\overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}\cdot d\sigma d\mathbf{r}'\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}$$

$$= \overset{+}{\rho} d\mathbf{r}'\cdot d\sigma\overset{+}{\boldsymbol{\beta}}\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}$$

(46) $$= \overset{+}{\boldsymbol{\beta}}\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}de.$$

Substituting $-\overset{+}{\mathbf{K}}\cdot\delta\mathbf{r}de$ for $\delta\overset{+}{U}$, (33) now becomes

(47) $$\int_{t_1}^{t_2}\{\overset{+}{\boldsymbol{\beta}}\times(\mathbf{H} - G\overset{+}{\mathbf{H}})\cdot\delta\mathbf{r}de + (\mathbf{E} - G\overset{+}{\mathbf{E}})\cdot\delta\mathbf{r}de + \overset{+}{\mathbf{K}}\cdot\delta\mathbf{r}de\}dt = 0,$$

from which we may infer that

$$\mathbf{E} + \overset{+}{\boldsymbol{\beta}} \times \mathbf{H} + \overset{+}{\mathbf{K}} - G(\overset{+}{\mathbf{E}} + \overset{+}{\boldsymbol{\beta}} \times \overset{+}{\mathbf{H}}) = 0. \tag{9}$$

Obviously, we may derive equation (10) by an exactly similar process in which the terms involving G do not enter. And if we wish to use an infinitesimal charge of some other shape, we may consider it as divided up into a number of cylinders, not necessarily right cylinders, such as we used above.[6]

Meanings of the Laws.— To find out what we can about the properties of the ether, we may now examine carefully the meanings of these five laws:

$$(1)\quad \nabla \cdot \overset{+}{\mathbf{E}} = \overset{+}{\rho}, \qquad (2)\quad \nabla \cdot \overset{-}{\mathbf{E}} = \overset{-}{\rho},$$

$$(3)\quad \nabla \times \overset{+}{\mathbf{H}} = \dot{\overset{+}{\mathbf{E}}} + \overset{+}{\rho}\overset{+}{\boldsymbol{\beta}}, \qquad (4)\quad \nabla \times \overset{-}{\mathbf{H}} = \dot{\overset{-}{\mathbf{E}}} + \overset{-}{\rho}\overset{-}{\boldsymbol{\beta}},$$

$$(11)\quad \delta \int_{t_1}^{t_2} \int\int\int_{\infty} \{(\mathbf{H}^2 - G\overset{+}{\mathbf{H}}^2) - (\mathbf{E}^2 - G\overset{+}{\mathbf{E}}^2 + 2U)\} dr dt = 0.$$

The first two of these laws contain no reference whatever to time, and deal with quantities whose existence is in no way dependent on motion or change with time. Therefore, we may infer that they probably express relations between the geometrical configurations of different parts of the ether, and show the dependence of these geometrical configurations upon the presence in the ether of the peculiar movable spots called charges, whose indestructibility and ability to be located definitely at different times (specified in equations (3) and (4), as well as the internal forces, suggest that they are due to the presence of some substances not present in the rest of the ether but freely movable through it. Since these substances can be located at any time if the vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$ are known at every point, the question arises whether any more information than the value of these vectors needs to be given to determine completely the configuration of the ether. A suggestion of the answer to this question is given by the fact that in applying Hamilton's Principle to problems of ordinary dynamics, the variations must be such as to give the actual configura-

[6] To be certain that no equations not derivable from equations (1)–(10) can be derived from (11) and (1)–(4), we need only to consider the facts that any possible variation in equation (11) can be made up of variations of the types treated above, and that the mutual energy of two independent variations of the first order is an infinitesimal of the second order.

tions exactly at the times t_1 and t_2, whereas, in equation (11) they must be such as to give the actual vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$. Hence, from analogy, we may say that these vectors are probably sufficient to specify the configuration of the ether completely.

And if this last statement is true, their time derivatives must be sufficient to specify completely, not only the quantities $\overset{+}{\boldsymbol{\beta}}$ and $\overset{-}{\boldsymbol{\beta}}$, but all the motions of the ether; and it seems probable that these motions at any point are specified by the values of $\dot{\overset{+}{\mathbf{E}}}$, $\dot{\overset{-}{\mathbf{E}}}$, $\overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}$, and $\overset{-}{\rho}\,\overset{-}{\boldsymbol{\beta}}$ at that point, and not by the values of the vectors $\overset{+}{\mathbf{H}}$ and $\overset{-}{\mathbf{H}}$, which depend on the values of the other vectors at distant points. This hypothesis is further strengthened by the fact that the whole theory of the ether might be developed without any use of these vectors, replacing $\overset{+}{\mathbf{H}}$ wherever it occurs by

$$\frac{1}{4\pi}\,\nabla\times\mathrm{Pot}\,(\dot{\overset{+}{\mathbf{E}}}+\overset{+}{\rho}\,\overset{+}{\boldsymbol{\beta}}),\quad\text{and}\quad\overset{-}{\mathbf{H}}\ \text{by}\ \frac{1}{4\pi}\,\nabla\times\mathrm{Pot}\,(\dot{\overset{-}{\mathbf{E}}}+\overset{-}{\rho}\,\overset{-}{\boldsymbol{\beta}}),$$

and therefore without any use of equations (3) and (4), except as they express the indestructibility of the charges.

Therefore we may consider equations (3) and (4) as merely equations of continuity and partial definitions of two convenient mathematical functions fully defined by equations (3) and (4) and (11) all together, and whose values at any point depend on the motions of the ether at all points, but not in any way on the motions or configurations at the point in question only. And thus, although they contain time derivatives and quantities dependent entirely on motion and existing only when there is motion, they tell us nothing about what is going to happen at some future time from what is happening now, and therefore cannot be considered as laws of motion, but only as mathematical definitions of convenient functions.

Equation (11), however, in form and substance, is essentially an equation of motion, from which no information about the geometrical configurations of the ether can be derived at any time, unless the configuration and motion at some other time, or the configurations at two other times, are specified; but without which no information about the configuration or motion at any time can be derived even if they are given at any number of other times.

Properties of the Ether. — The first question that arises about the properties of the ether is, Is its structure continuous or granular?

To answer this question definitely seems impossible, but at any rate, we can say that if it is granular and if these equations are to hold, the structure must be exceedingly minute compared to the dimensions of the electrons. A further suggestion is given by the fact that in the geometrical equations, (1) and (2), the positive and negative quantities appear very similar, but seem to be more or less independent of each other; while in the equation of motion (11), and in the phenomena of vacuum tube discharges, etc., differences between the actions of the positive and negative quantities appear, that seem to show that not only are the electrons of the different signs made up differently, but that the forces are transmitted by more or less independent, and slightly different, structures in the medium. As this condition of affairs seems to be incompatible with the idea of a continuous medium we are thereby led to the conception of a medium in which there are probably two similar, but slightly different, interlacing, granular structures, whose grains and distances between them are inconceivably small, even compared to the electrons.

The question of solid or fluid character of the ether appears easier to answer; for if it were fluid, that is, if no amount of shear at any point would change the properties at that point in such a way as to affect the subsequent motions around it, a transverse wave would be impossible. And if it were quasi-elastic, with effects analogous to viscosity, that would enable it to transmit wireless telegraph waves as well as the shortest known light waves, electrostatic forces around stationary charges should be due to some effect entirely different from that which produces those of the wireless wave, so that slow continued flow of ether might occur without hindrance. But the changes of electric force near a moving electron may be much more rapid than those of the wireless wave, and yet there appears to be no viscous retardation of its motion. Furthermore, the aberration of light and experiments such as that of H. A. Wilson[7] on the polarization of a dielectric cylinder rotating in a magnetic field seem to show that no flow of ether occurs in moving matter. These considerations and many others compel us to reject the fluid theory, and to say that the structures of the ether are solid. But by "solid" we must not mean possessed of ordinary solid elasticity, but merely that every particle is permanently connected to the particles near it by connections that cannot be deformed indefinitely, or even by a finite amount without affecting the subsequent motion.

7 H. A. Wilson. "Electric Effect of Rotating a Dielectric in a Magnetic Field," Roy. Soc. Proc., **73**, pp. 490–492. June 22, 1904.

As we must not assume ordinary elasticity, so also we must not assume ordinary inertia of the fundamental particles. For, after all, Newton's laws of motion, that we observe for ordinary matter, appear to be only approximations to the laws that result from equation (11), the more general law of motion. And furthermore, they are by no means the only ones consistent with the relative nature of time and space, nor is there any other *a priori* philosophical reason for assuming that they are true, while there is good philosophical reason for assuming that Hamilton's Principle, the mathematical expression of the perfect efficiency of the fundamental machinery of nature, is at least plausible. Therefore, whatever motions of the parts of the ether it may involve, and whether or not it is easy for us, with our Newtonian mechanical training, to form a mental picture of the dynamics of these motions, the fundamental law of the dynamics of the ether, or of any mental picture of it, must be Hamilton's Principle.

A Model of the Ether. — To get a mental picture of the actions of the ether, we must now make some arbitrary assumptions as to the nature of the two interlacing structures and the strains in them that are represented by the vectors $\overset{+}{\mathbf{E}}$ and $\overset{-}{\mathbf{E}}$. For simplicity we may think of them as nets with cubical meshes with each knot of either net in the centre of a mesh of the other, wherever the electric vectors are zero. The vector $\overset{+}{\mathbf{E}}$ may be a very minute displacement of one of these nets from this position, and the vector $\overset{-}{\mathbf{E}}$ the negative of a similar displacement of the other. If we now suppose the strings of these nets to be hollow and rigid, and the knots to be hollow boxes, so constructed that the displacements of the nets will be those of an incompressible substance, we may suppose an electric charge to be a region in which the pipes and boxes of one of the nets are filled with a liquid of high surface tension, that will expand the boxes into which it flows, and cause a divergence of the displacement of the net. An electron will then be a region of this sort, in the shape of a hollow sphere when at rest, of which every dimension, including the thickness, is very large compared with the meshes of the net. The pipes and boxes of that net that lie inside this region may be filled with a fluid whose only properties are adhesion with everything it touches and a constant hydrostatic tension, independent of its volume. For the connections between the nets we may assume anything we please.

Equations (1) and (2) are satisfied by this model, which also gives an interesting interpretation for (3) and (4). For in free ether the

vector $\overset{+}{\mathbf{H}}$ becomes a hydrokinetic flow-function for the motion of the positive net; and where there is any positive charge it is a flow-function for the motion of the net plus that of the charge. Similarly the vector $\overset{-}{\mathbf{H}}$ is the negative of a flow-function of the motion of the negative net and charges. And in each case, equations (5) and (6) tell us that it is the solenoidal flow-function that is required.

The equation of motion is, as we expected, one which we have some difficulty in applying. But if we split it up into equations (5) to (10), and then combine them properly, we may use in electrical problems only the vectors **E** and **H**, representing the *relative* displacement of the positive net from the negative, and the flow-function of the *relative* motion. And in gravitational problems the vectors **E** and **H** disappear entirely.

Collisions of Electrons. — An interesting application of this model is to the problem of collisions of electrons, of the same or opposite signs, as in the case of a cathode particle striking an electron in the metal it hits. If they are of the same kind they will evidently become flattened as they come together. But as soon as they are within about their own length of each other, the side of either of them nearest the other will be effected not only by the displacement due to the presence of the other, but also by the displacements radiated from the other on account of its acceleration. To make the vectors balance, as required by equations (9) and (10), its acceleration must therefore be so much greater than that required by the inverse square law that they can never collide.

In the case of two electrons of different kinds, both are lengthened, and they come together faster than the inverse square law would demand. But since they may go right through each other perfectly freely, there need not be any of the destructive effects that one might expect from other theories.

Retarded Potentials.[8] — In calculating the values of the retarded potentials due to moving electrons it is found necessary to treat each electron as if its charge were not the same as when at rest, but changed in the ratio $(1 - \beta_r)^{-1}$, where β_r is the component of $\boldsymbol{\beta}$ in the direction towards the point at which we wish to know the potential. This has been interpreted by some writers [9] as indicating that all electromagnetic actions are due to some sort of pulsation of the electrons, and are

[8] For information about retarded potentials, see Lorentz, "Theory of Electrons," Chap. 1.

[9] L. de la Rive, Phil. Mag., **13**, p. 279.

stronger if the pulsations are more rapid, so that the Doppler effect is introduced if the charge is moving. But with the model it is obvious that any such interpretation is unnecessary; for the important quantity is not the actual charge of the electron, but the volume of the ether in which there was a spreading of the net at such a time as to affect the point in question at the time in question.

Summary.

Because of the apparently absolute nature of acceleration, as well as for other reasons, we find it necessary to assume the existence of the ether, and therefore desirable to learn as much as possible of its properties. To do this, we first reduce the laws of all its phenomena, including gravitation and the relativity-principle, to five equations, and then examine their meanings; and find that two of them are probably laws of the geometrical configurations of the different parts of the ether; two more, equations partially defining two convenient vectors, and stating the indestructibility of electricity; while the fifth, Hamilton's Principle, is a law of motion, expressing the perfectly efficient cooperation of the different parts of the fundamental mechanism of the universe.

From these laws we may draw certain conclusions about the structure and properties of the ether, which are not, however, enough to enable us to determine exactly what it is. But by a few simple assumptions, we obtain an imaginable model of its actions. And since the model is based directly on the electromagnetic laws, it may be applied, without fear of error, to any electromagnetic problem, to enable us to obtain a qualitative result without mathematical analysis.